① Show $\chi(G) \leq \Delta(G)+1$

We'll do induction on $|V(G)|$

Base: a single vertex colored with one color, $d(v)=0 \Rightarrow 1 \leq 0+1$

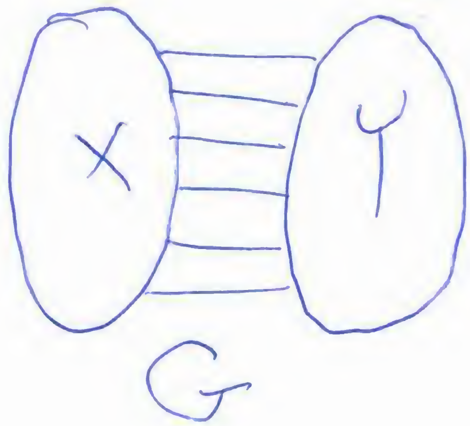
I.H.: Assume for some $P(k)$ $k>1$ and graph H, $|V(H)|=k$ that $\chi(H) \leq \Delta(H)+1$

I.S.: - Consider $P(n)=G$ $n>k$

- Consider $v \in V(G) : d(v)=\Delta(G)$
- Consider $H=G-v$
- I.H. on H, H can be colored in $\Delta(H)+1$ colors, $\Delta(H) \leq \Delta(G)$
- Add v back into G, in the worst case, each vertex in $N(v)$ has a different color as given in the coloring on H, we assign $c(v)=\Delta(G)+1=d(v)+1$
- We now have a proper $(\Delta(G)+1)$-coloring on G □

(2) - All bipartite graphs are 2-colorable

- We can maximize the degrees by constructing a complete bipartite graph



$\forall x \in X, \forall y \in Y : (x,y) \in E(G)$

$|X| = |Y| +/- 1$

$\forall v \in V(G) : d(v) \geq \frac{|V|-1}{2}$

(3) - We note that non-empty graphs can be bounded below by $\chi(G) \geq 2$

- As we saw with (1), general graphs can be bounded above with $\chi(G) \leq \Delta(G) + 1$
- However, (2) shows us that the bound given in (1) is arbitrarily loose. I.e, we can have a graph G where $|V(G)| \to \infty$ and $\Delta(G) \to \infty$ yet $\chi(G) = 2$ can be fixed.

④ show $\chi(K_n, k) = k(k-1)\dots(k-n+1)$

Basis: K_1 obviously can be colored with k colors

$$\chi(K_1, k) = k$$

I.H.: Assume for $P(k) = K_n$ that

$$\chi(K_n, k) = k(k-1)\dots(k-n+1)$$

I.S.: Show for K_{n+1}

- We add new vertex to K_n and connect it to all existing vertices to create K_n
- this new vertex can be colored with any color that doesn't show up on K_n
- $(k-n)$ different ways
- I.H. on $K_n \Rightarrow \chi(K_n, k) = k(k-1)\dots$
- $\chi(K_{n+1}, k) = k(k-1)\dots(k-n+1)(k-n)$
- $m = n+1$

so $\chi(K_m, k) = k(k-1)\dots(k-m+2)(k-m+1)$

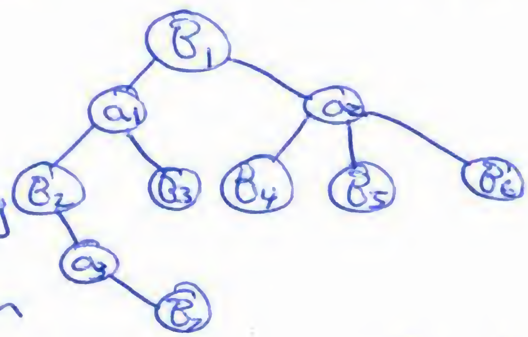
$\square$

(5) - Consider block-cutpoint graph of G

- Each block is an even or odd cycle $\chi(B_i) = 2$ or 3

- Block-cutpoint graph is a tree

- Consider greedy coloring starting at some block, fully coloring it then proceeding to neighboring blocks in a BFS/DFS-like fashion



- Note: the coloring of each block can be done independently of child B_i and only depends on a single articulation point => the maximum chromatic number of the graph is just the maximum chromatic number of any block

$$\Rightarrow \boxed{2 \le \chi(G) \le 3}$$